

Energy in Yang-Mills on a Riemann Surface

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Abstract

Sengupta's lower bound for the Yang-Mills action on smooth connections on a bundle over a Riemann surface generalize to the space of connections whose action is finite. In this larger space the inequality can always be saturated. The Yang-Mills critical sets correspond to critical sets of the energy action on a space of paths. This may shed light on Atiyah and Bott's conjecture concerning Morse theory for \mathcal{A}/\mathcal{G} .

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I Introduction

One approach [1, 2] to quantum Yang-Mills on a Riemann surface of genus g requires rewriting the Yang-Mills action in terms of the energy of a $2g$ -tuple of paths in the symmetry group G (This assumes $g \geq 1$. For $g = 0$, the energy is that of a based loop in G .) The energy of such paths appears more recently in Yang-Mills inequalities Sengupta has developed [3].

Sengupta considers the space of smooth connections, grouped into subspaces by certain requirements on holonomy. For each subspace, there is a loop in G whose energy bounds from below the Yang-Mills action on that subspace. For appropriate choices of the requirements on holonomies, this lower bound can be saturated; Yang-Mills connections are precisely those which saturate this bound.

Uhlenbeck [4] has shown that, in two dimensions, the space of connections whose Yang-Mills action is finite contains discontinuous connections. Theorem III.1 below provides a lower bound for the Yang-Mills action on this larger space. It is analogous to Sengupta's, but in this space the bound can always be saturated. One might then suppose that Yang-Mills connections arise when these saturating connections are also smooth; this is the import of Proposition III.1.

These relations between the Yang-Mills action and the energy of paths may help answer a question raised in Atiyah and Bott's seminal work [5] on the topology of the moduli space of Yang-Mills connections; namely, does the Yang-Mills action, which they show to be equivariantly perfect, in fact define a Morse stratification? Theorem III.2 describes the correspondence between the critical sets of the Yang-Mills action and those of the energy on the relevant space of paths, for which there is reason to believe the analytic issues are more tractable.

II The Geometry of $\mathcal{A}/\mathcal{G}_m$

To describe the required energy requires some background on the structure of the quotient $\mathcal{A}/\mathcal{G}_m$ of the space of connections modulo gauge transformations. Here \mathcal{A} refers to connections with finite total curvature on a given G -bundle P over a Riemann surface Σ , and \mathcal{G}_m refers to the space of gauge transformations which are the identity at a specified point $m \in \Sigma$. What follows is an overview of the essential elements; details are in the references [1, 2].

Let D , a regular $4g$ -gon be a fundamental domain for Σ , chosen so that m corresponds to the center of D . The edges making up ∂D represent the generators $\{a_i, b_i\}_{i=1}^g$ of $\pi_1(\Sigma)$, and are identified in pairs, with opposite orientations, as in Figure 1.

Theorem 3.1 of [2] states that $\mathcal{A}/\mathcal{G}_m$ is itself a principal fiber bundle over $Path^{2g} G$ with an affine-linear fiber. Here $Path^{2g} G$ is the space of $2g$ -tuples of paths in G subject

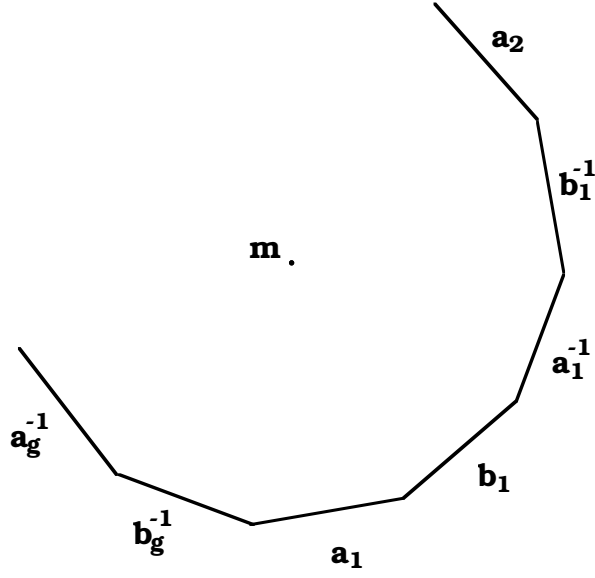


Figure 1: The fundamental domain D

to a single relation on the $4g$ endpoint values of the paths. There is an obvious energy function (see Eq 2) on this base space $Path^{2g} G$; its critical points are precisely the images of Yang-Mills connections. To understand how this arises, it will suffice to examine the projection $\xi : \mathcal{A}/\mathcal{G}_m \rightarrow Path^{2g} G$.

Consider holonomies by a given connection A about the following loops in Σ : Pick polar coordinates (r, θ) on D centered at m . For a given point p of the edge $a_1 \subset \partial D$, the radial path from m to that point followed by the radial path back to m from the corresponding point p^{-1} of a_1^{-1} defines a loop in Σ . See Figure 2. Relative to a fixed choice of basepoint in the fiber over m , the holonomy by A about this loop determines an element of G . Now, let the point p vary within a_1 . The corresponding holonomies trace out a path α_1 in G . Holonomies about radial paths through the points of the other edges $b_1, a_2, b_2, \dots, a_g, b_g$ similarly determine paths $\beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g$. Taken together, these define the $2g$ -tuple $\vec{\gamma}_A = (\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$. These $2g$ paths are not completely independent of each other, however, as the radii to the vertices of ∂D each lie on two distinct loops in Σ whose holonomies define the endpoint values of distinct paths in G . In fact, traversing, in the appropriate order, each such radius out to the vertex and back again to m gives a certain product of the endpoint values of the paths in $\vec{\gamma}_A$. On the other hand, by construction, the holonomy about this path must be the identity in G . Equating these gives the relation defining $Path^{2g} G$:

$$\alpha_1(0)\beta_1(1)^{-1}\alpha_1(1)^{-1}\beta_1(0) \cdots \alpha_g(0)\beta_g(1)^{-1}\alpha_g(1)^{-1}\beta_g(0) = \mathbf{1}.$$

Define $\xi([A]) \equiv \vec{\gamma}_A$. This is well-defined on $\mathcal{A}/\mathcal{G}_m$, since acting on A by an element of \mathcal{G}_m has no effect on $\vec{\gamma}_A$. Clearly, adding to A a Lie-algebra-valued one-form τ which vanishes in the radial directions of D also has no effect on $\vec{\gamma}_A$. In fact, in $\mathcal{A}/\mathcal{G}_m$, as

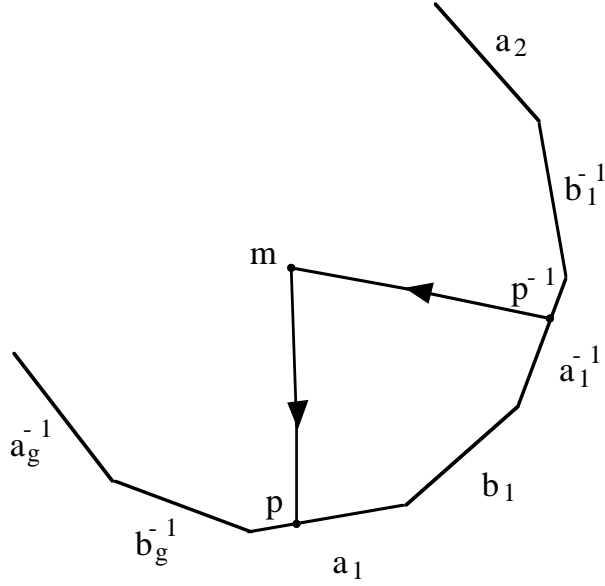


Figure 2: A radial path in D

a bundle over $Path^{2g} G$, the fiber over $\vec{\gamma}_A$ is the space $\{[A + \tau] : \tau|_{\text{radii}} = 0\}$. This, and the fact that ξ is onto, is proven in Theorem 3.1 of [2].

If the bundle P is not topologically trivial, then, as detailed in [6], its topology is determined by an element z of the center of the universal cover \widehat{G} of G . On lifting P to a \widehat{g} -bundle, the space $Path^{2g} G$ is replaced by the corresponding space for \widehat{G} with the relation $\alpha_1(0)\beta_1(1)^{-1}\alpha_1(1)^{-1}\beta_1(0) \cdots \alpha_g(0)\beta_g(1)^{-1}\alpha_g(1)^{-1}\beta_g(0) = z$. Henceforth, though we omit the hats, we assume we are on the lifted bundle with the corresponding relation.

III The Yang-Mills action

Consider now the restriction of the Yang-Mills action on $\mathcal{A}/\mathcal{G}_m$ to the fiber through $\vec{\gamma}_A$.

$$S([A]) = \langle F_A, F_A \rangle,$$

where the inner product combines the invariant inner product on the Lie algebra, the metric-induced inner product on forms at each point and integration over Σ . Along the fiber, $F_{A+\tau} = F_A + D_A\tau$, since the term quadratic in τ vanishes. Thus,

$$S([A + \tau]) = S([A]) + 2 \langle F_A, D_A\tau \rangle + \langle D_A\tau, D_A\tau \rangle.$$

Theorem 4.2 of [2] ensures that the requirement $\langle F_A, D_A\tau \rangle = 0$, for every τ vanishing along radii, singles out a unique choice for a continuous connection \tilde{A} to serve as an

“origin” in the fiber. Note that $[\tilde{A}]$ defines a section of $\mathcal{A}/\mathcal{G}_m$ over $Path^{2g} G$. Relative to this choice of origin,

$$S([\tilde{A} + \tau]) = S([\tilde{A}]) + \langle D_A \tau, D_A \tau \rangle. \quad (1)$$

(For τ of the specified form, $D_{\tilde{A}} \tau = D_A \tau$.) The key point is that $S([\tilde{A}])$ pulls back to the energy of $\vec{\gamma}_A$. This follows from the condition on $F_{\tilde{A}}$ which implies directly that $*F_{\tilde{A}}$ is covariantly constant along radii. Thus, in $S([\tilde{A}]) = \langle F_{\tilde{A}}, F_{\tilde{A}} \rangle$, $F_{\tilde{A}}$ may be replaced by its average along the radius. This, however, by a non-Abelian analog of Stoke’s theorem, or by Polyakov’s formula, is $\alpha_i^{-1} \dot{\alpha}_i$ (or $\beta_i^{-1} \dot{\beta}_i$) for some i depending on the value of θ . In fact, for an appropriate choice of parametrization, determined by the area element on Σ ,

$$S([\tilde{A}]) = \frac{1}{2} \sum_{i=1}^{2g} \|\dot{\gamma}_i\|^2 \equiv E(\vec{\gamma}_A), \quad (2)$$

as detailed in Section 5.1 of [2]. Here γ_i denotes the i th component of $\vec{\gamma}_A$. For a generic connection, which must be gauge equivalent to $\tilde{A} + \tau$, Eq 1 thus becomes

$$S([\tilde{A} + \tau]) = E(\vec{\gamma}_A) + \langle D_A \tau, D_A \tau \rangle \quad (3)$$

It leads immediately to a lower bound on the Yang-Mills action on a given fiber:

Theorem III.1 *For any connection A representing an element of the fiber through*

$$\vec{\gamma}_A \in Path^{2g} G,$$

$$S([A]) \geq \frac{1}{2} \sum_{i=1}^{2g} \|\gamma_i\|^2,$$

with equality holding iff A agrees with the section \tilde{A} (up to gauge transformation).

Proof: This is an immediate consequence of Eq 3, since the second term on the right-hand side is positive semi-definite, and zero iff $\tau = 0$. \square

Given this decomposition of the Yang-Mills action, it is easy to see how its critical points correspond directly to critical points of the energy E .

Theorem III.2 *The connection \tilde{A} represents a Yang-Mills critical point iff $\vec{\gamma}_{\tilde{A}}$ is a critical point of the energy E .*

Proof: Suppose $A = \tilde{A} + \tau$ is a Yang-Mills critical point; that is, a point at which $S([A + \tau])$ is stationary. (There is no loss of generality in omitting a possible gauge transformation on one side of this equation.) By considering just τ of the form $\tau = t\tau_0$, for $t \in R$, it is clear from Eq 3 that $\tau = 0$ is a necessary condition for A to be a critical point. It then follows that $\tilde{\gamma}_A$ must be a critical point of the energy. The converse is immediate. \square

To relate this picture, in which connections need not be smooth and the energy bound can always be saturated, to Sengupta's, in which connections must be smooth and the energy bound can only be saturated on the fibers containing Yang-Mills connections, note that in the fibers over critical points of the energy the connection \tilde{A} is smooth.

Proposition III.1 *If $\vec{\gamma}$ is a critical point of E , then the corresponding \tilde{A} is smooth.*

Proof: A simple calculus of variations computation shows that $\vec{\gamma}$ extremizes E iff

$$\frac{\partial}{\partial \theta} \gamma_i^{-1} \dot{\gamma}_i = 0.$$

On the other hand, this condition also ensures that the covariantly constant curvatures $F_{\tilde{A}}$, related by the non-Abelian analog of Stokes Theorem mentioned previously, are continuous at m . This was the only place \tilde{A} might have failed to be smooth. \square

IV A possible application

Atiyah and Bott suggest equivariant Morse theory might apply to the cohomology of the Yang-Mills moduli space, and, more particularly, their stratification may correspond to the Morse stratification for the Yang-Mills action. With this in mind, they prove the Yang-Mills action is an equivariantly perfect Morse function. However, analytic concerns prevent them from developing the theory more fully, except in genus zero. There Bott and Samuelson [7] have shown that \mathcal{A}/\mathcal{G} is topologically equivalent to based loops in G , and that Morse theory arguments go through for a wide variety of symmetric spaces including these based loops.

The geometric picture of $\mathcal{A}/\mathcal{G}_m$ as an affine-linear bundle shows it is topologically equivalent to its base space $Path^{2g} G$. Passing from \mathcal{G}_m to \mathcal{G} , this becomes $Path^{2g} G/G$, where a given element $g \in G$ acts adjointly on each path: $\gamma_i(t) \mapsto$

$g^{-1}\gamma_i(t)g$. Moreover Eq 3 says the section $[\tilde{A}]$ pulls the Yang-Mills action back to the energy on $Path^{2g} G/G$. Clearly, the Morse theory for this base space, if such exists, would be the Morse theory for \mathcal{A}/\mathcal{G} . Moreover, the generality of Bott and Samuelson's results is nearly sufficient to apply them directly to $Path^{2g} G/G$. The endpoint condition, however, requires careful treatment, which we defer to future work.

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